JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 4, July–August 2008

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Dynamic Control Allocation for Tracking Time-Varying Control Demand

W. C. Arun Kishore,* S. Sen,† and G. Ray‡
Indian Institute of Technology, Kharagpur 721 302, India
and
T. K. Ghoshal§
Jadvapur University, Kolkata 700 032, India

DOI: 10.2514/1.34085

I. Introduction

MODERN aircraft, missiles, and launch vehicles are fitted with more control surfaces/actuators than controlled variables to achieve fast maneuvers. These extra effectors provide a certain degree of redundancy and hence can be used to achieve multiple control objectives. There are several combinations of actuator positions which produce the same virtual control effort demanded by the high level controller and hence achieve desired performance. Control allocation of such overactuated systems is generally formulated as an optimization problem to minimize energy loss, power consumption, and other similar costs, subject to constraints such as actuator saturation in rate and position, and a specified set of dynamics for operation in the linear range. Extensive surveys that compare and identify the limitations of control allocation algorithms are presented in [1,2]. Among these, weighted norm minimization [3–5] provides an explicit solution, but without taking actuator saturation into consideration. This methodology has been extended in [6] to accommodate the saturation constraints by selecting a weight matrix in a linear matrix inequality (LMI) formulation, but it is only suitable for a limited region of control demand space. Linear programming, mixed integer, and quadratic programming approaches have also been applied to solve the allocation problems formulated as weighted norm minimization and convex quadratic optimal allocation subject to linear equality and inequality constraints[7–9].

The literature cited in the preceding paragraph considered the control allocation problem for cases where the dynamics of actuators are much faster than the system dynamics. Oppenheimer and Doman [10,11] have shown that actuator dynamics have a significant effect on control allocation and suggested a method to compensate for actuator dynamics; however, the results presented were only for first- and second-order dynamics. The dynamic control allocation methods presented in [12,13] used an actuator model to predict the desired actuator state and control input trajectory, and thereby determined allocation as a solution to an optimal trajectory tracking problem. But such a method needs a prediction module to predict control demand, actuator state, and output trajectory, which in turn requires a receding control input horizon ahead of time. Obtaining a control input horizon ahead of time is not always practical, which restricts the algorithm to trajectory tracking problems.

The algorithm proposed in this paper combines actuator dynamics compensation with allocation and generalizes it to actuators of any arbitrary order, and does not require any control horizon ahead of time. In this paper, we consider overactuated systems, where each individual actuator has linear dynamics, and whose deflections are bounded in amplitude and rate. The allocation problem is cast into an LMI problem, taking into account the actuator dynamics, and thereby providing a straightforward methodology to solve. Details of dynamic control allocation are provided in Sec. II. This method is validated through a numerical example in Sec. III. A representative missile problem has been considered in Sec. IV to justify the necessity of including actuator dynamics along with position and rate saturation constraints, and it has been shown that neglecting the actuator dynamics may lead to unstable operation, even though the actuator bandwidth is larger than the system bandwidth.

II. Dynamic Control Allocation

Control allocation plays a vital role in overactuated systems, particularly when the actuators have saturation limits, different effectiveness, and dynamics. Also, when the allocation is separated from the controller, reconfiguration of control due to actuator failure can be attained from the allocation algorithm without redesign of the control strategy. The proposed control allocation scheme is designed to generate commands to deflect the actuators, and to achieve desired moments or accelerations demanded by the controller. A conceptual block diagram of the control strategy is shown in Fig. 1.

The control law for an overactuated m input, p output system ($m > p$), is generally designed for virtual inputs $u_{\mathrm{des}}(t) \in \mathbb{R}^p$, the desired moment/acceleration demanded by the controller; $\delta_{\mathrm{cmd}} \in \mathbb{R}^m$ is the commanded deflection to the actuator and $u_{\mathrm{act}} \in \mathbb{R}^p$ is the actual moment/acceleration that is going to the plant due to the actual control effector deflection. The actual moment $u_{\mathrm{act}} \in \mathbb{R}^p$ is assumed to be a mapping of the actual deflection of the actuator $\delta_{\mathrm{act}} \in \mathbb{R}^m$ through a nonlinear function $g(t, \delta_{\mathrm{act}})$. The actuator deflection is constrained to

$$\delta_{\min} \le \delta_{\mathrm{act}} \le \delta_{\max} \tag{1}$$

where $\delta_{\min}$ and $\delta_{\max}$ are the lower and upper saturation limit, and rate is limited by

$$\underline{\dot{\delta}} \le \dot{\delta}_{\mathrm{act}} \le \bar{\dot{\delta}} \tag{2}$$

and whose motion is otherwise governed by linear dynamics.

The flight control law is considered to operate as a discrete-time system with a sampling period of T, and the rate of change of actuator response is approximated by a first-order difference approximation [6] as

Received 17 August 2007; accepted for publication 18 January 2008.

*Research Scholar, Department of Electrical Engineering; akishore@ee.iitkgp.ernet.in.

†Professor, Department of Electrical Engineering; ssen@ee.iitkgp.ernet.in.

‡Professor, Department of Electrical Engineering; gray@ee.iitkgp.ernet.in.

§Professor, Department of Electrical Engineering; tkghoshal@gmail.com.

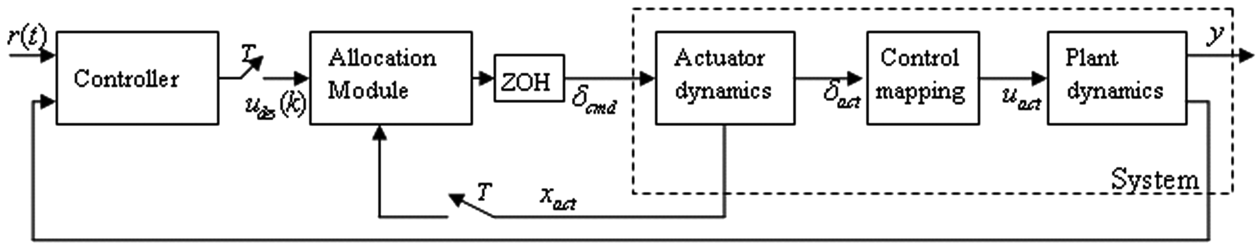


Fig. 1 Structure of control allocation problem.

$$\dot{\delta}_{\mathrm{act}} = \frac{\delta_{\mathrm{act}}(t+T) - \delta_{\mathrm{act}}(t)}{T} \tag{3}$$

Which means rate saturation constraints can be represented as

$$\underline{\dot{\delta}} \leq \frac{\delta_{\mathrm{act}}(t+T) - \delta_{\mathrm{act}}(t)}{T} \leq \bar{\dot{\delta}} \tag{4}$$

or

$$\delta_{\mathrm{act}}(t) + \underline{\dot{\delta}}T \leq \delta_{\mathrm{act}}(t+T) \leq \delta_{\mathrm{act}}(t) + \bar{\dot{\delta}}T \tag{5}$$

Thus, rate saturation is approximated as a limitation in the change of deflection on each time interval. Combining the rate saturation constraint with the actual amplitude saturation limit, the saturation limit for the next interval can therefore be defined as the most restrictive of the two constraints, i.e.,

$$\begin{aligned} \underline{\delta}(t+T) &= \max[\delta_{\min}, \delta_{\mathrm{act}}(t) + \underline{\dot{\delta}}T] \\ \bar{\delta}(t+T) &= \min[\delta_{\max}, \delta_{\mathrm{act}}(t) + \bar{\dot{\delta}}T] \end{aligned} \tag{6}$$

Static control allocation schemes consider the control effectiveness mapping $g(t, \delta_{\mathrm{act}}) = G(t)\delta_{\mathrm{act}}$, which reduces the complexity of the allocation problem and also neglects actuator dynamics (i.e., $\delta_{\mathrm{cmd}} = \delta_{\mathrm{act}}$), and assumes that actuators are much faster than the dynamics of controlled elements. In contrast, the proposed algorithm applies to systems to achieve u_{des} through optimal allocation, where dynamics of actuators have a significant effect on system performance.

A. Actuator Dynamics

There are m number of actuators, each having linear dynamics. The order of the jth actuator is n_j, $j = 1, 2, \ldots, m$.

Assumption 1: The dynamics of the actuators are known, time invariant, and perfectly uncoupled. The state-space representation of the overall actuator system dynamics is described as

$$\dot{x}_{\mathrm{act}} = A_{\mathrm{act}} x_{\mathrm{act}} + B_{\mathrm{act}} \delta_{\mathrm{cmd}} \qquad \delta_{\mathrm{act}} = C_{\mathrm{act}} x_{\mathrm{act}} \tag{7}$$

where $x_{\mathrm{act}} \in \mathbb{R}^N$, actuator states $A_{\mathrm{act}} \in \mathbb{R}^{N\times N}$, $B_{\mathrm{act}} \in \mathbb{R}^{N\times m}$, $C_{\mathrm{act}} \in \mathbb{R}^{m\times N}$, and

$$N = \sum_{j=1}^{m} n_j$$

From assumption 1, Eq. (7) has a block diagonal structure $A_{\mathrm{act}} = \mathrm{diag}([A_{\mathrm{act},1}, A_{\mathrm{act},2}, \ldots, A_{\mathrm{act},m}])$, $B_{\mathrm{act}} = \mathrm{diag}([B_{\mathrm{act},1}, B_{\mathrm{act},2}, \ldots, B_{\mathrm{act},m}])$, and $C_{\mathrm{act}} = \mathrm{diag}([C_{\mathrm{act},1}, C_{\mathrm{act},1}, \ldots, C_{\mathrm{act},1}])$.

The discrete equivalent of the dynamic actuator system becomes

$$x_{\mathrm{act}}(k+1) = Ax_{\mathrm{act}}(k) + B\delta_{\mathrm{cmd}}(k) \qquad \delta_{\mathrm{act}}(k) = Cx_{\mathrm{act}}(k) \tag{8}$$

where A, B, and C are actuator system parameter matrices of proper dimension in the discrete domain with sampling time T. The control effectiveness mapping $G(t)\delta_{\mathrm{act}}(t)$ maps the actual deflection of actuators to the actual control input developed. The desired control moment for tracking $r(k)$ is $u_{\mathrm{des}}(k)$, and actual moment achieved is

$$u_{\mathrm{act}}(k) = G(k)\delta_{\mathrm{act}}(k) \tag{9}$$

The controller, in Fig. 1, provides the desired control moments/forces $u_{\mathrm{des}}(k)$ from the reference input $r(k)$ and system response. This control demand is available to the allocation module at the kth instant to determine $\delta_{\mathrm{cmd}}(k)$.

Assumption 2: (A, B) is controllable and (A, C) is observable for all $k \geq 0$.

Assumption 3: For the discrete-time model of the actuator, matrix CB is nonsingular and hence the effect of $\delta_{\mathrm{cmd}}(k)$ is reflected in $\delta_{\mathrm{act}}(k+1)$.

The actuator dynamics produce a delay of one sampling instant, to respond to the demanded control input [i.e., $u_{\mathrm{des}}(k)$ appears at the input of the plant at $(k+1)$th instant]. The commanded deflections $\delta_{\mathrm{cmd}}(k)$ should achieve the desired forces and moment [Eq. (10)].

$$\begin{aligned} u_{\mathrm{des}}(k) &= u_{\mathrm{act}}(k+1) = G(k+1)\delta_{\mathrm{act}}(k+1) \\ \text{or} \quad u_{\mathrm{des}}(k) &= G(k+1)CAx_{\mathrm{act}}(k) + G(k+1)CB\delta_{\mathrm{cmd}}(k) \end{aligned} \tag{10}$$

The effectiveness mapping $G(k)$ is assumed to be known ahead of time. There may exist an infinite number of combinations of $\delta_{\mathrm{cmd}}(k)$ satisfying Eq. (10). Among these, a solution is selected by optimizing an additional criterion that minimizes the net actuator deflection (this is roughly equivalent to minimizing the drag for aerodynamic control).

B. Dynamic Allocation Algorithm to Finite Bandwidth Actuators

Dynamic control allocation with finite bandwidth actuators can be formulated as an optimization problem to minimize the performance measure

$$J_1 = \delta_{\mathrm{act}}(k+1)^T W_\delta^2 \delta_{\mathrm{act}}(k+1) \tag{11}$$

where $W_\delta = \mathrm{diag}([w_{\delta,1} w_{\delta,2} w_{\delta,3} \ldots w_{\delta,m}])$ is positive definite, and each $w_{\delta,i}$ specifies the relative weight. J_1 is minimized subject to the constraints in Eqs. (6), (8), and (10). If the control demand is unattainable due to position or rate saturation of the actuators, this formulation may lead to an infeasible problem. To alleviate such a possibility, the equality constraint [Eq. (10)] is relaxed and a new performance measure is selected by heavily penalizing an additional term related to moment error as shown:

$$\begin{aligned} J = {} & \delta_{\mathrm{act}}(k+1)^T W_\delta^2 \delta_{\mathrm{act}}(k+1) \\ & + [u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)]^T W_u^2 [u_{\mathrm{des}}(k) \\ & - G(k+1)\delta_{\mathrm{act}}(k+1)] \end{aligned} \tag{12}$$

where $W_u = \mathrm{diag}([w_{u,1} w_{u,2} w_{u,3} \ldots w_{u,p}])$ is positive definite. Components of W_u are selected relatively high compared with those of W_δ to reduce allocation error. The minimum of the function is less than $\gamma > 0$ such that

$$\gamma - J > 0 \tag{13}$$

or

$$\begin{aligned} & \gamma - \delta_{\mathrm{act}}(k+1)^T W_\delta^2 \delta_{\mathrm{act}}(k+1) \\ & - [u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)]^T W_u^2 [u_{\mathrm{des}}(k) \\ & - G(k+1)\delta_{\mathrm{act}}(k+1)] > 0 \end{aligned} \tag{14}$$

The nonlinear convex inequality in Eq. (14) is converted to LMI form using Schur complements. An LMI that is equivalent to Eq. (14) is given by

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{15}$$

where $Q(x) = Q(x)^T, R(x) = R(x)^T$, and $S(x)$ are affine functions of matrix variable x. The Schur complement of this is given by [14]

$$R(x) > 0, \qquad Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \tag{16}$$

which, along with Eq. (14), leads to the matrix inequality

$$\begin{bmatrix} \gamma & [\delta_{\mathrm{act}}(k+1)]^T W_\delta & [u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)]^T W_u \\ W_\delta \delta_{\mathrm{act}}(k+1) & I & 0 \\ W_u[u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)] & 0 & I \end{bmatrix} > 0 \tag{17}$$

The actual deflection of the actuators is constrained to Eq. (6), which can be rewritten as

$$e_i\delta_{\mathrm{act}}(k+1) - e_i\underline{\delta}(k+1) > 0, \qquad i = 1, 2, \ldots, m \qquad e_j\bar{\delta}(k+1) - e_j\delta_{\mathrm{act}}(k+1) > 0, \qquad j = 1, 2, \ldots, m \tag{18}$$

Where e_i and e_j are unit vectors along $\delta_{\mathrm{act},i}$ and $\delta_{\mathrm{act},j}$. Combining Eqs. (17) and (18), the control allocation problem becomes the solution of the LMI constrained minimization problem given by Eq. (19):

$$\begin{aligned} &\min_{\delta_{\mathrm{cmd}}(k)} \gamma \\ &\text{Subject to} \begin{bmatrix} \gamma & [\delta_{\mathrm{act}}(k+1)]^T W_\delta & [u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)]^T W_u \\ W_\delta \delta_{\mathrm{act}}(k+1) & I & 0 \\ W_u[u_{\mathrm{des}}(k) - G(k+1)\delta_{\mathrm{act}}(k+1)] & 0 & I \end{bmatrix} > 0 \\ &\gamma > 0 \\ &e_i\delta_{\mathrm{act}}(k+1) - e_i\underline{\delta}(k+1) > 0, \qquad i = 1, 2, \ldots, m \\ &e_j\bar{\delta}(k+1) - e_j\delta_{\mathrm{act}}(k+1) > 0, \qquad j = 1, 2, \ldots, m \\ &\text{where} \\ &x_{\mathrm{act}}(k+1) = Ax_{\mathrm{act}}(k) + B\delta_{\mathrm{cmd}}(k) \\ &\delta_{\mathrm{act}}(k+1) = Cx_{\mathrm{act}}(k+1) \\ &\underline{\delta}(k+1) = \max[\delta_{\min}, \delta_{\mathrm{act}}(k) + \underline{\dot{\delta}}T] \\ &\bar{\delta}(k+1) = \min[\delta_{\max}, \delta_{\mathrm{act}}(k) + \bar{\dot{\delta}}T] \end{aligned} \tag{19}$$

This LMI formulation in Eq. (19) is amenable to solution by standard algorithms. The allocation uses the current instant control demand to determine actuator command which, when applied to the actuator dynamics, takes one sample time for the command to appear at the actuator output. A delay of one sampling time is inevitable.

Solution of Eq. (19) is critical for real-time application to flight control problems. The optimization problem is to be solved iteratively. However, the LMI in Eq. (19) forms a convex optimization problem and the convergence is guaranteed if a feasible solution exists. Fast convergence is generally expected if the allocation at the previous instant is taken as the feasible starting point. The algorithm in the present case is tested with MATLAB LMI toolbox. With a fast processor and dedicated software developed for the particular application, it is expected that the proposed methodology should be applicable to online control allocation as well.

C. Dynamic Allocation for Infinite Bandwidth Actuators

Control allocation of infinite bandwidth actuators with position and rate limits is a special case of the dynamic allocation formulated previously. The combined position and rate constraints given by Eq. (6), and restated in Eq. (18), fixes the possible deflection of the actuators at the end of the current interval. The dynamic allocation command to the actuator after neglecting dynamics is stated as an LMI formulation in Eq. (20). Here, the commanded deflection is assumed to appear instantaneously [i.e., $\delta_{\mathrm{cmd}}(k) = \delta_{\mathrm{act}}(k) = \delta(k)$] at the output of the actuator because it has sufficiently large bandwidth, compared with that of the system dynamics.

$$\begin{aligned} &\min_{\delta(k)} \gamma \\ &\text{Subject to} \begin{bmatrix} \gamma & \delta(k)^T W_\delta & [u_{\mathrm{des}} - G\delta(k)]^T W_u \\ W_\delta \delta(k) & I & 0 \\ W_u[u_{\mathrm{des}} - G\delta(k)] & 0 & I \end{bmatrix} > 0 \\ &\gamma > 0 \\ &e_i\delta(k) - e_i\underline{\delta}(k) > 0, \quad i = 1, 2, \ldots, m \\ &e_j\bar{\delta}(k) - e_j\delta(k) > 0, \quad j = 1, 2, \ldots, m \\ &\text{where} \\ &\underline{\delta}(k) = \max[\delta_{\min}, \delta_{\mathrm{act}}(k-1) + \underline{\dot{\delta}}T] \\ &\bar{\delta}(k) = \min[\delta_{\max}, \delta_{\mathrm{act}}(k-1) + \bar{\dot{\delta}}T] \end{aligned} \tag{20}$$

III. Numerical Example

The dynamic allocation algorithm discussed in this paper is tested with a numerical example from [11]. A set of four rate and position constrained dynamic actuators having different dynamics is considered. Allocation to the actuator is performed by solving Eq. (19). The control effectiveness mapping function $G(k)$ considered here is time invariant, and for all k

$$G(k) = \begin{bmatrix} -0.4 & 0.4 & -0.1 & 0.1 \\ -0.1 & -0.1 & -0.6 & -0.6 \\ -0.1 & 0.1 & -0.1 & 0.1 \end{bmatrix} \tag{21}$$

The elements of $G(k)$ have units of $\mathrm{rad/s^2/deg}$ (i.e., vehicle acceleration in $\mathrm{rad/s^2}$ per degree effector deflection). The dynamics of the actuators are

$$\begin{aligned} \frac{\delta_{\mathrm{act}_1}(s)}{\delta_{\mathrm{cmd}_1}(s)} &= \frac{2.5(s+10)}{s^2+7.071s+25}, \\ \frac{\delta_{\mathrm{act}_2}(s)}{\delta_{\mathrm{cmd}_2}(s)} &= \frac{2.5(s+10)}{s^2+7.071s+25} \\ \frac{\delta_{\mathrm{act}_3}(s)}{\delta_{\mathrm{cmd}_3}(s)} &= \frac{49}{s^2+7s+49}, \\ \frac{\delta_{\mathrm{act}_4}(s)}{\delta_{\mathrm{cmd}_4}(s)} &= \frac{5}{s+5} \end{aligned} \tag{22}$$

and the positions and rates are limited by

$$\begin{aligned} \delta_{\min} &= [-1.5 \quad -1.5 \quad -1.5 \quad -1.5], \ \mathrm{deg} \\ \delta_{\max} &= [0.4 \quad 1.5 \quad 1.5 \quad 1.5], \ \mathrm{deg} \\ \underline{\dot{\delta}} &= [-10 \quad -10 \quad -10 \quad -3], \ \mathrm{deg/s} \\ \bar{\dot{\delta}} &= [10 \quad 10 \quad 10 \quad 3], \ \mathrm{deg/s} \end{aligned} \tag{23}$$

These actuator limits cause at least one actuator to reach saturation in position and rate for the given control demand. The control demand signals selected for allocation are given by

$$u_{\mathrm{des}}(t) = \begin{bmatrix} 0.15\sin(0.3\pi t^2 + \pi t) \\ 0.3\sin(0.3\pi t^2 + \pi t) \\ 0.15\sin(0.3\pi t^2 + \pi t) \end{bmatrix} \mathrm{rad/s^2} \tag{24}$$

The sampling interval and weight matrices selected for simulation are $T = 0.01$ s, $W_\delta = \mathrm{diag}([1 \quad 1 \quad 1 \quad 1])$, and $W_u = \mathrm{diag}([100 \quad 100 \quad 100])$, respectively.

The control demand allocated using the proposed algorithm, implemented using the MATLAB LMI toolbox, verifies the effectiveness of the algorithm when the actuators reach saturation. Simulation results in Figs. 2 and 3 show that the algorithm is able to achieve the demanded control from the actuator output (i.e., $u_{\mathrm{des}}(t) \cong G\delta_{\mathrm{act}}(t+T) \ \forall \ t$), even though actuator 1 saturates in position and actuator 4 saturates in rate. Near ideal performance is achieved. The delay of one sampling interval is due to the discrete representation of the actuator dynamics. Oppenheimer and Doman [11] have obtained similar performance with a compensation scheme separate from allocation.

IV. Implementation to Missile Control

Having successfully applied the proposed method for an overactuated allocation problem in the previous section, the technique is now applied for a vertically or a near-vertically launched thrust vector controlled missile. This is also applicable to vertically launched satellite launch vehicles. The missile has a core and three strap-on boosters, separated by 120 deg with eight actuators to control the nozzle deflection, $\delta_1, \delta_2, \delta_3, \ldots, \delta_8$, as shown in Fig. 4. Arrows represent the positive direction of deflections of the engines.

The control moments developed due to engine deflections is proportional to the offset of nozzle gimbal point from the center of mass of missile and nozzle deflection in radians. The control effectiveness of actuators are mapped through Eq. (9) where

$$G(k) = \begin{bmatrix} 0.55 & 0 & 0.1299 & 0.075 & 0.075 & 0.1299 & 0.15 & 0 \\ 0 & 0.55 & -0.075 & 0.1299 & -0.1299 & 0.075 & 0 & 0.15 \\ 0 & 0 & -0.15 & 0 & 0 & 0.15 & 0 & -0.15 \end{bmatrix} \tag{25}$$

The missile dynamic model is obtained by considering undamped weathercocking angular motion and neglecting aerodynamic damping. The transfer function $G_1(s)$ in Fig. 5 relates attitude rates to applied moments in Eq. (26), and the system outputs are the three attitudes of pitch, yaw, and roll.

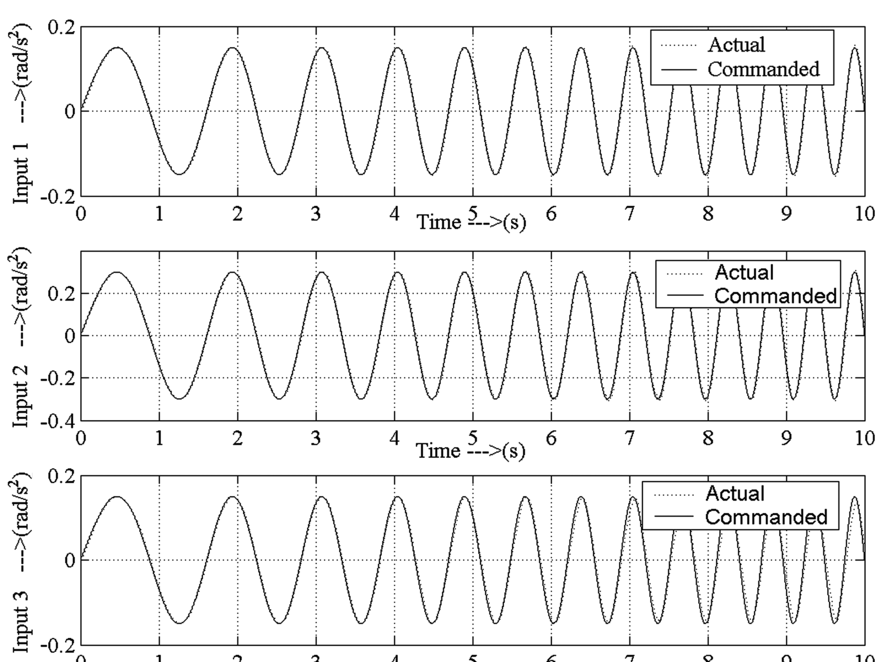


Fig. 2 Control demand to be generated by actuators.

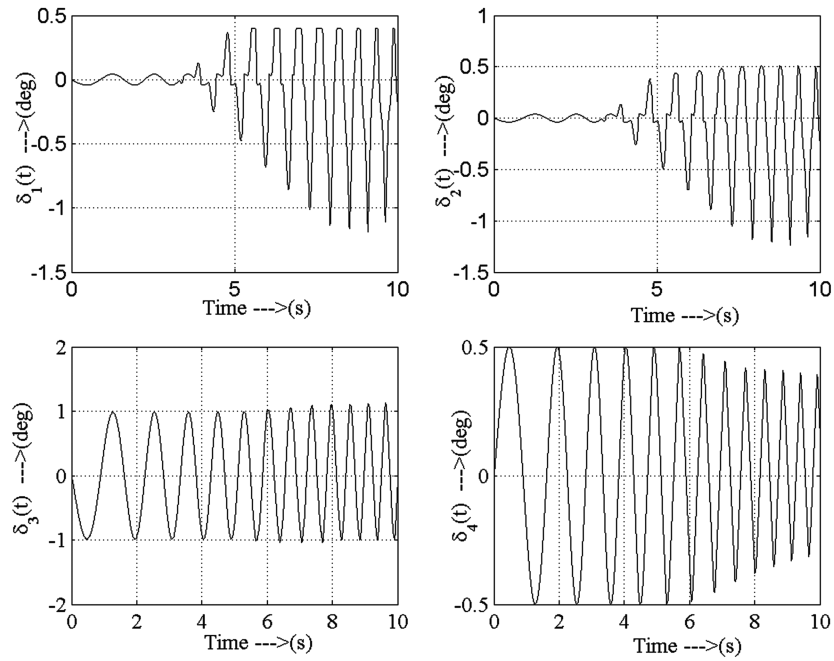
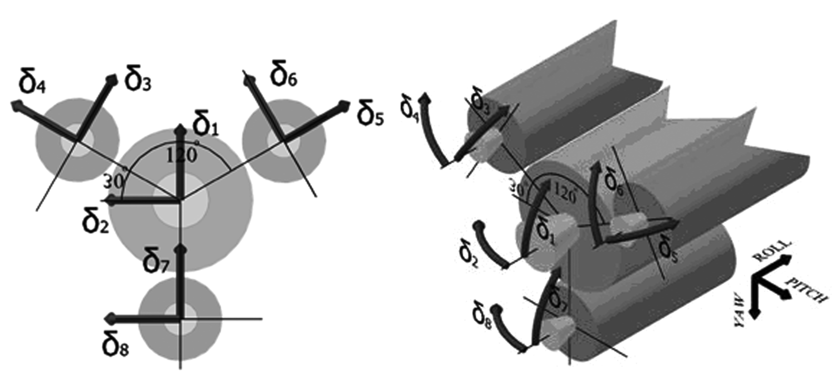


Fig. 3 Deflections of individual actuators.



Rear view **Isometric view**

Fig. 4 Actuator configurations.

$$G_1(s) = \frac{Q(s)}{U_{\text{act}}(s)} = \begin{bmatrix} \frac{200(s+1)}{s^2+100} & 0 & 0 \\ 0 & \frac{200(s+1)}{s^2+100} & 0 \\ 0 & 0 & \frac{200}{s} \end{bmatrix} \tag{26}$$

The attitude control block and rate control block are separately shown in Fig. 5 for clarity. Because a number of actuators with different effectiveness and dynamics had to be dealt with, along with variable control allocation, the autopilots were designed iteratively. This started with selecting the inner loop gain K_2, neglecting actuator dynamics, and using a nominal allocation. To accommodate actuator lags and outer loop contribution, the inner loop was designed with liberal margins without considering actuator dynamics. The gain crossover frequencies were chosen at about one-third of the smallest actuator natural frequency. The outer loop gain K_1 was chosen to obtain the desired closed loop bandwidth.

The dynamics of actuators are uncoupled. The actuator system is designed to have second-order dynamics with a damping ratio of 0.7, natural frequencies of 10 Hz for actuators 1 and 2 (core), and 20 Hz for the remaining actuators (boosters). A sampling time of 0.01 s is used to simulate the system. The combined eight-actuator system is on the order of 16 with parameter matrices

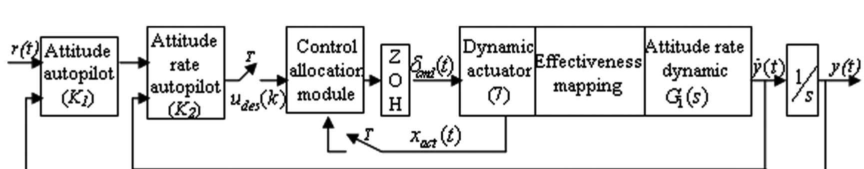


Fig. 5 Block schematic of missile system.

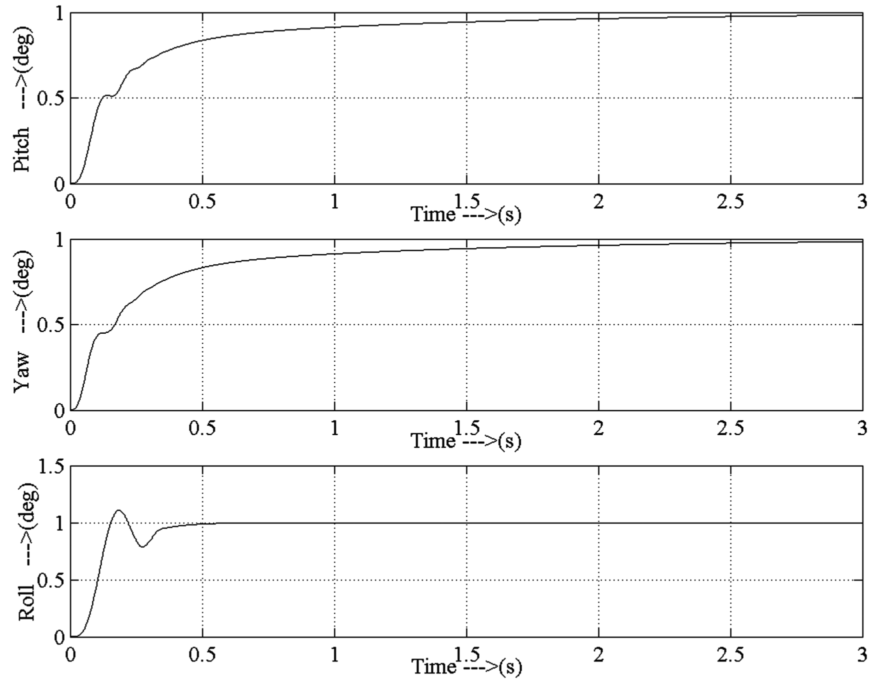


Fig. 6 Attitude response to step input.

$$
\begin{aligned}
A_{\mathrm{act},j} &= \begin{bmatrix} 0.3065 & -24.5852 \\ 0.0062 & 0.8543 \end{bmatrix} \quad \text{for } j = 1, 2 \\
A_{\mathrm{act},j} &= \begin{bmatrix} -0.0592 & -57.0755 \\ 0.0036 & 0.5767 \end{bmatrix} \quad \text{for } j = 3, 4, \ldots, 8 \\
B_{\mathrm{act},j} &= \begin{bmatrix} 0.0062 \\ 3.69e-5 \end{bmatrix} \quad \text{for } j = 1, 2 \\
B_{\mathrm{act},j} &= \begin{bmatrix} 0.0036 \\ 2.68e-5 \end{bmatrix} \quad \text{for } j = 3, 4, \ldots, 8 \\
C_{\mathrm{act},j} &= [0 \quad 3948] \quad \text{for } j = 1, 2 \\
C_{\mathrm{act},j} &= [0 \quad 15791] \quad \text{for } j = 3, 4, \ldots, 8
\end{aligned} \tag{27}
$$

The position and rate of change of the actuator deflections are bounded by Eq. (28)

$$
\begin{aligned}
\delta_{\max} &= [2 \quad 2 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6]^T \ \text{deg} \\
\delta_{\min} &= -[2 \quad 2 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6]^T \ \text{deg} \\
\bar{\dot{\delta}} &= [20 \quad 20 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60]^T \ \text{deg/s} \\
\underline{\dot{\delta}} &= -[20 \quad 20 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60 \quad 60]^T \ \text{deg/s}
\end{aligned} \tag{28}
$$

A. Step Response with Dynamic Actuator

The system is simulated using MATLAB with a step command of 1 deg applied to pitch, yaw, and roll simultaneously. The weight

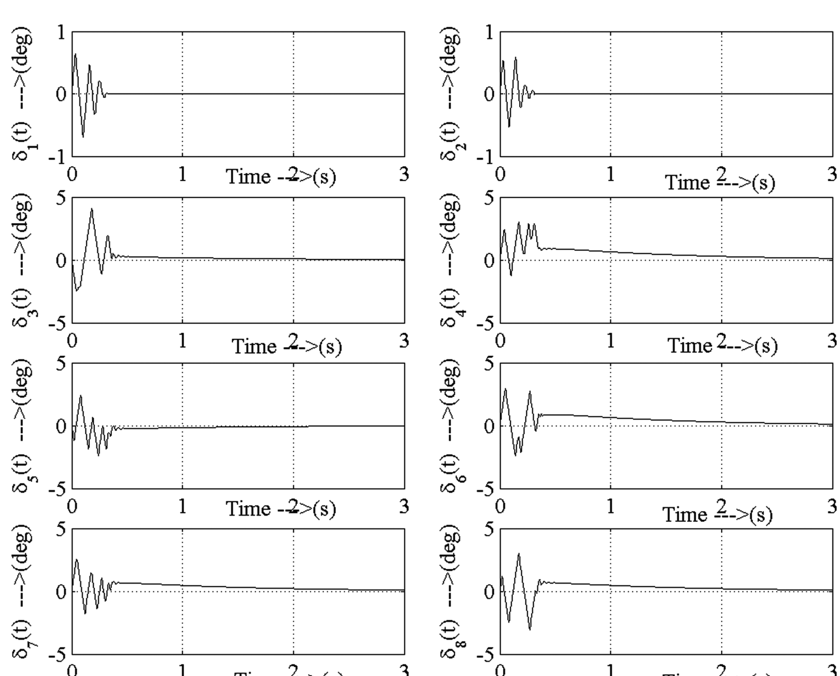


Fig. 7 Allocation to the eight actuators on dynamic allocation.

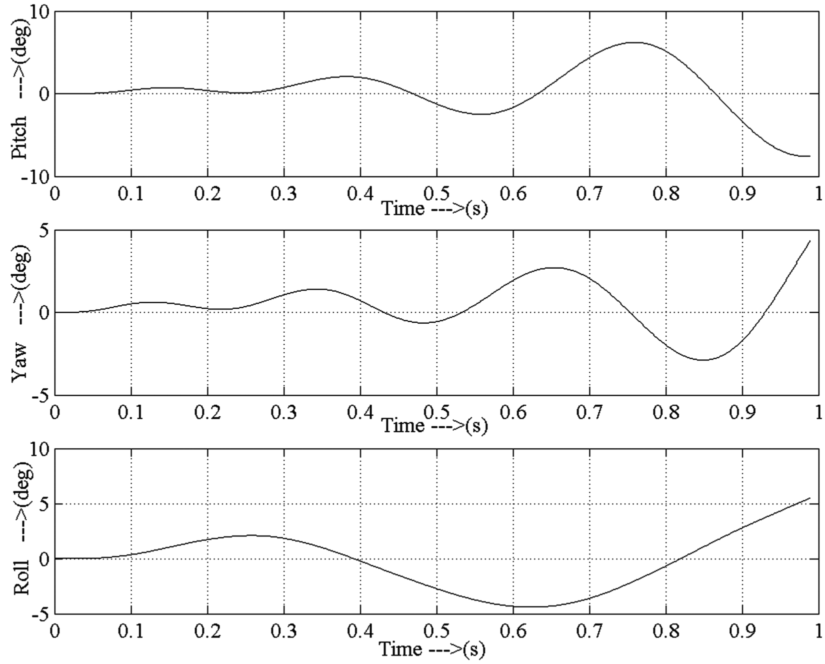


Fig. 8 System responses to step command for allocation without considering actuator dynamics.

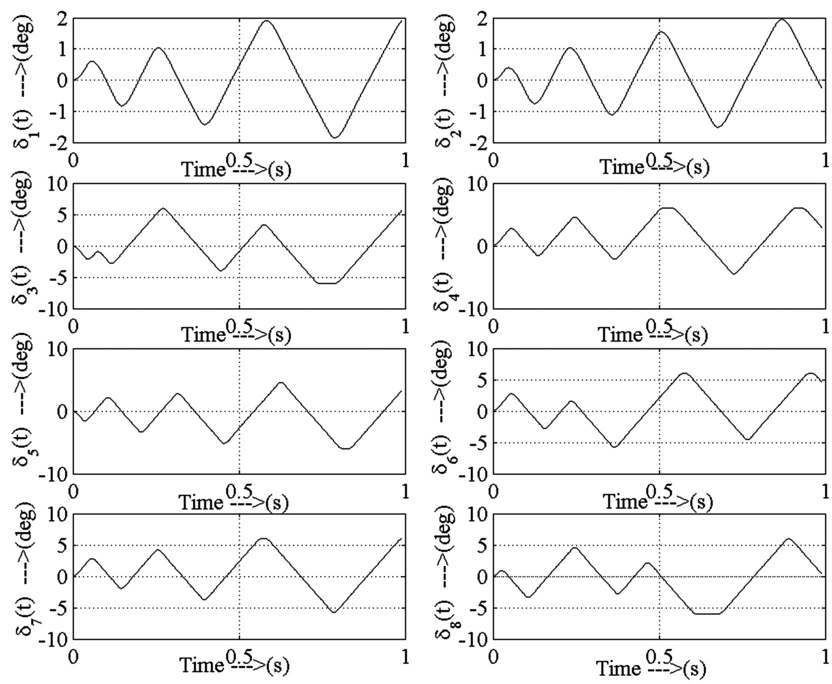


Fig. 9 Actuator responses to step command for allocation without considering actuator dynamics.

matrix $W_\delta = \mathrm{diag}([100 \quad 100 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1])$ is selected to share maximum control contribution from faster boosters and less from the core, $W_u = \mathrm{diag}([100 \quad 100 \quad 100])$ is selected to heavily penalize moment errors in the control allocation cost. Attitude and rate autopilot gains $K_1 = \mathrm{diag}([5 \quad 5 \quad 8])$ and $K_2 = \mathrm{diag}([0.5908 \quad 0.5908 \quad 0.25])$ are selected so as to satisfy the performance specifications. The responses obtained are shown in Figs. 6 and 7, where control allocation to the actuators is obtained by solving Eq. (19) using the MATLAB LMI toolbox. The proposed algorithm stabilizes the system despite actuator rate saturation and provides an acceptable response. The nonsmooth response observed during transients on Fig. 7 is due to the actuator rate saturation.

B. Allocation Neglecting Actuator Dynamics

The simulation results in Figs. 8 and 9 show unstable responses. This instability is a result of neglecting actuator dynamics when formulating the control allocation problem for the previously considered missile. The responses are obtained by posing the control allocation problem as Eq. (20), rather than Eq. (19).

This result verifies the necessity of incorporating actuator dynamics into the control allocation problem. Thus, applying the proposed dynamic control allocation improves the system stability and performance characteristics.

V. Conclusions

A dynamic control allocation algorithm for an overactuated system has been proposed, which takes into account the effects of actuator dynamics and constraints on actuator position and rate. The proposed control allocation scheme is based on the solution of an optimization problem, via an LMI formulation, which provides a systematic design technique and distributes control authority among different types of actuators, resulting in stable tracking of a time-varying control demand. The advantages of the proposed algorithm are that it provides commands to actuators that account for actuator dynamics of arbitrary order. A simulation study conducted on an attitude control problem achieved significant improvement over the traditional methods of static control allocation algorithm in presence of realistic actuator dynamics.

Acknowledgments

The authors are grateful to the editor and the anonymous reviewers for their comments that helped to improve the quality of the paper.

References

[1] Bodson, M., "Evaluation of Optimization Methods for Control Allocation," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 4, 2002, pp. 703–711.

[2] Enns, D., "Control Allocation Approaches," AIAA Paper 1998-4109, Aug. 1998.

[3] Kishore, A. W. C., Sen, S., and Ray, G., "Disturbance Rejection and Control Allocation of Over-Actuated Systems," *IEEE International Conference on Industrial Technology*, Inst. of Electrical and Electronics Engineers, New York, Dec. 2006, pp. 1054–1059.

[4] Härkegard, O., "Dynamic Control Allocation Using Constrained Quadratic Programming," *Journal of Guidance, Control, and Dynamics*, Vol. 27. No. 6, 2004, pp. 1028–1034.
doi:10.2514/1.11607

[5] Härkegard, O., and Glad, S. T., "Resolving Actuator Redundancy-Optimal Control vs Control Allocation," *Automatica*, Vol. 41, No. 1, 2005, pp. 137–144.
doi:10.1016/S0005-1098(04)00255-9

[6] Bošković, J.D., Ling, B., Prasanth, R., and Mehra, R. K., "Design of Control Allocation Algorithms for Over-Actuated Aircraft Under Constraints Using LMIs," *Proceedings of the 41st IEEE Conference on Decision and Control*, Inst. of Electrical and Electronics Engineers, New York, Dec. 2002, pp. 1711–1716.

[7] Bolender, M. A., and Doman, D. B., "Nonlinear Control Allocation Using Piecewise Linear Functions: A Linear Programming Approach," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 3, 2005, pp. 558–563.
doi:10.2514/1.12997

[8] Bolender, M. A., and Doman, D. B., "Nonlinear Control Allocation Using Piecewise Linear Functions," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 6, 2004, pp. 1017–1027.
doi:10.2514/1.9546

[9] Simmons, A. T., and Hodel, A. S., "Control Allocation for the X-33 Using Existing and Novel Quadratic Programming Techniques," *Proceedings of the American Control Conference*, Vol. 2, American Automatic Control Council, Evanston, IL, June 2004, pp. 1701–1706.

[10] Oppenheimer, M. W., and Doman, D. B., "Methods for Compensating for Control Allocator and Actuator Interactions," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 5, 2004, pp. 922–927.
doi:10.2514/1.7004

[11] Oppenheimer, M. W., and Doman, D. B., "Method for Compensation of Interactions Between Second-Order Actuators and Control Allocators," *Proceedings of the IEEE Aerospace Conference*, Inst. of Electrical and Electronics Engineers, New York, 2005, pp. 1–8.

[12] Luo, Y., Serrani, A., Yurkovich, S., Doman, D. B., and Oppenheimer, M. W., "Model Predictive Dynamic Control Allocation with Actuator Dynamics," *Proceedings of the American Control Conference*, Vol. 2, American Automatic Control Council, Evanston, IL, June 2004, pp. 1695–1700.

[13] Luo, Y., Serrani, A., Yurkovich, S., Doman, D. B., and Oppenheimer, M. W., "Model-Predictive Dynamic Control Allocation for Reentry Vehicles," *Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 1, 2007, pp. 100–113.
doi:10.2514/1.25473

[14] Boyd, S., Feron, E., Ghaoui, L. E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1994, pp. 7–31.